# Discreteness-Aware AMP for Reconstruction of Symmetrically Distributed Discrete Variables

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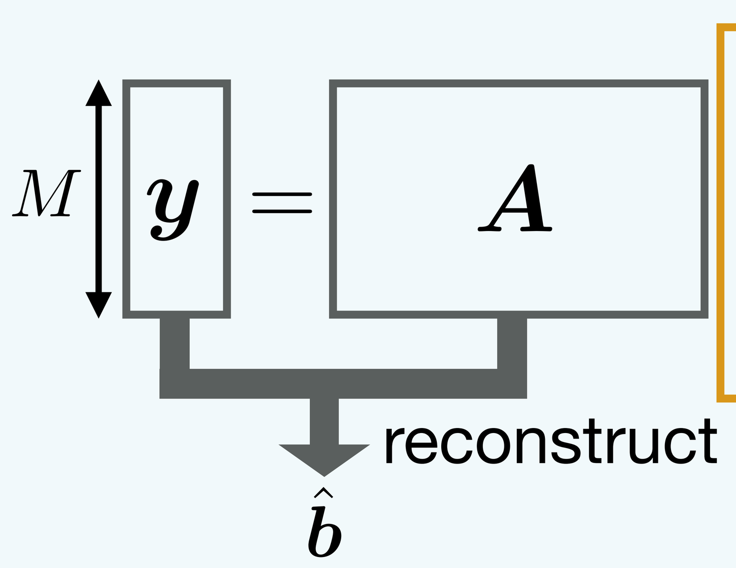
## Abstract

We propose a message passing-based algorithm to **reconstruct a discrete-valued vector** whose elements have a symmetric probability distribution. The proposed algorithm, referred to as **discreteness-aware approximate message passing (DAMP)**, borrows the idea of the AMP algorithm for compressed sensing. We analytically evaluate the performance of DAMP via state evolution framework to **derive a required number of linear measurements** for the exact reconstruction with DAMP.

## 1. Introduction

**Discrete-valued vector reconstruction**

reconstruct a discrete-valued vector $\boldsymbol{b} \in \mathbb{R}^N$
from its linear measurements $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{b} \in \mathbb{R}^M$ $(N \geq M)$




**Potential applications**

- multiuser detection in M2M communication (Machine-to-Machine)
- MIMO signal detection (Multiple-Input Multiple-Output)
- FTN signaling (Faster-than-Nyquist)

**Purpose of this work**

- propose a **low-complexity algorithm** for the discrete-valued vector reconstruction
- **theoretically analyze** the performance of the proposed algorithm via state evolution [1]

## 2. Proposed DAMP Algorithm

Assumption

$\boldsymbol{b} \in \{0, \pm r_1, \ldots, \pm r_L\}^N$ $(\Pr(b_j = r_\ell) = \Pr(b_j = -r_\ell))$

$\boldsymbol{A}$ is composed of i.i.d. variables with zero mean and variance $1/M$

**SOAV (Sum-of-Absolute-Value) optimization [2]**

use the fact that some elements of $\boldsymbol{b} \pm r_\ell \boldsymbol{1}$ are $0$

$$\hat{\boldsymbol{b}} = \arg\min_{\boldsymbol{x} \in \mathbb{R}^N} \left\{ q_0 \|\boldsymbol{x}\|_1 + \sum_{\ell=1}^{L} q_\ell (\|\boldsymbol{x} - r_\ell \boldsymbol{1}\|_1 + \|\boldsymbol{x} + r_\ell \boldsymbol{1}\|_1) \right\}$$
$$\text{subject to } \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

parameter ($q_\ell$)

apply the idea of AMP algorithm [1]

**Proposed algorithm (DAMP: Discreteness-aware AMP)**

① Initialization: $\boldsymbol{x}^{-1} = \boldsymbol{x}^0 = \boldsymbol{0}, \ \boldsymbol{z}^{-1} = \boldsymbol{0}$

② For $t = 0, 1, \ldots,$ calculate

estimate of $\boldsymbol{b}$ ($\boldsymbol{x}^{t+1}$); parameter ($\lambda$)

$$\boldsymbol{x}^{t+1} = \eta(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{z}^t + \boldsymbol{x}^t, \lambda\sigma_t),$$
$$\boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \frac{1}{\Delta}\boldsymbol{z}^{t-1}\langle \eta'(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{z}^{t-1} + \boldsymbol{x}^{t-1}, \lambda\sigma_{t-1})\rangle$$

complexity: $O(MN)$ per iteration

$\Delta = M/N$: observation ratio

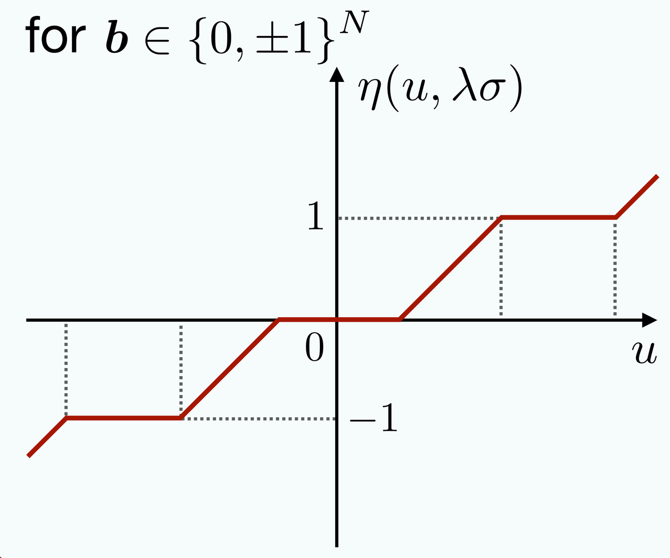
$\langle\cdot\rangle$: mean

$\sigma_t^2 = \frac{1}{N}\|\boldsymbol{x}^t - \boldsymbol{b}\|_2^2$: Mean-Square-Error (MSE)*

*In practice, we use the estimate of $\sigma_t^2$

**soft thresholding function**

for $\boldsymbol{b} \in \{0, \pm 1\}^N$




## 3. State Evolution for DAMP

**State evolution [1]**: predict the behavior of the MSE $\{\sigma_t^2\}_{t=0,1,\ldots}$ in the large system limit $(N, M \to \infty, M/N = \Delta)$

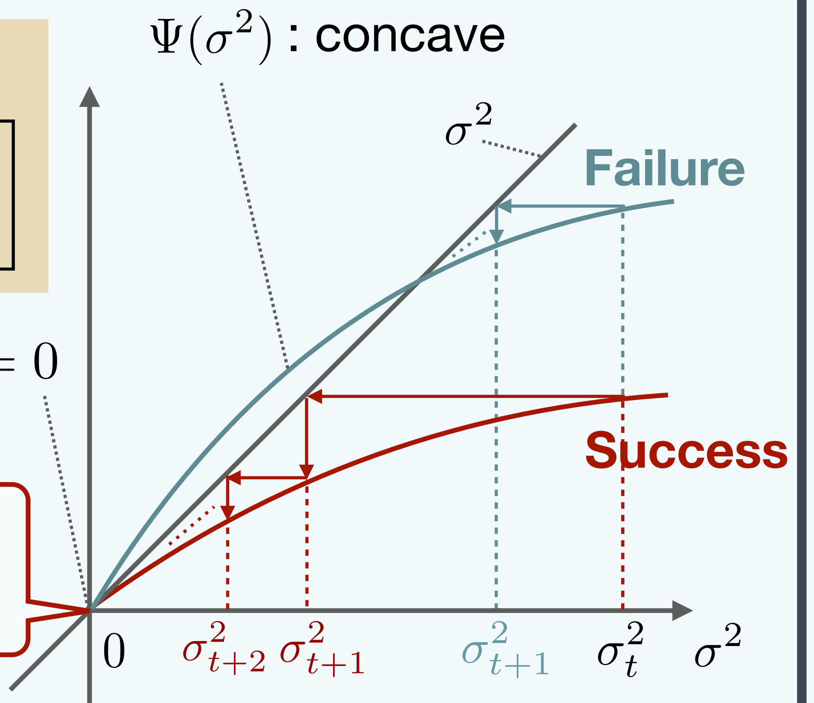
$$\sigma_{t+1}^2 = \Psi(\sigma_t^2)$$
$$\Psi(\sigma^2) = \mathrm{E}\left[\left\{\eta\left(X + \frac{\sigma}{\sqrt{\Delta}}Z, \lambda\sigma\right) - X\right\}^2\right]$$

$X \sim$ distribution of $b_j$
$Z \sim \mathcal{N}(0, 1)$

$\Psi(\sigma^2)$: concave

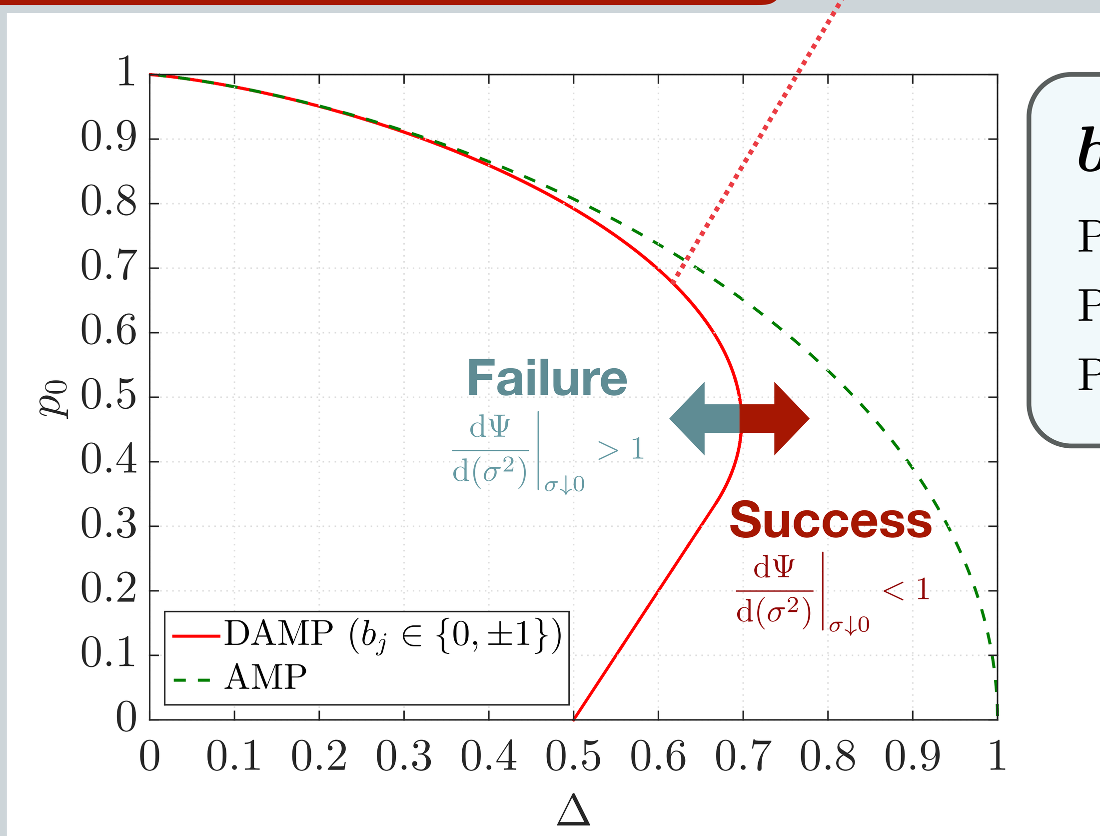
$\Psi(0) = 0$

$$\left.\frac{\mathrm{d}\Psi}{\mathrm{d}(\sigma^2)}\right|_{\sigma \downarrow 0} < 1 \Rightarrow \text{Success}$$



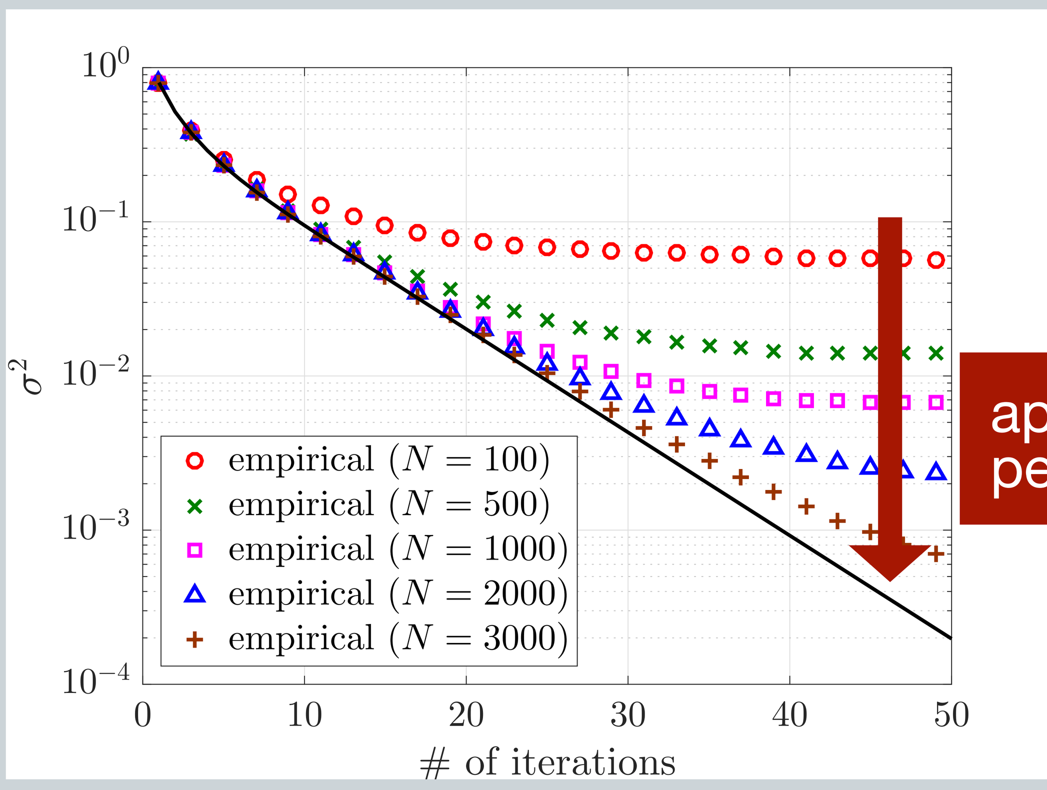

**Phase transition of DAMP**

required observation ratio for DAMP



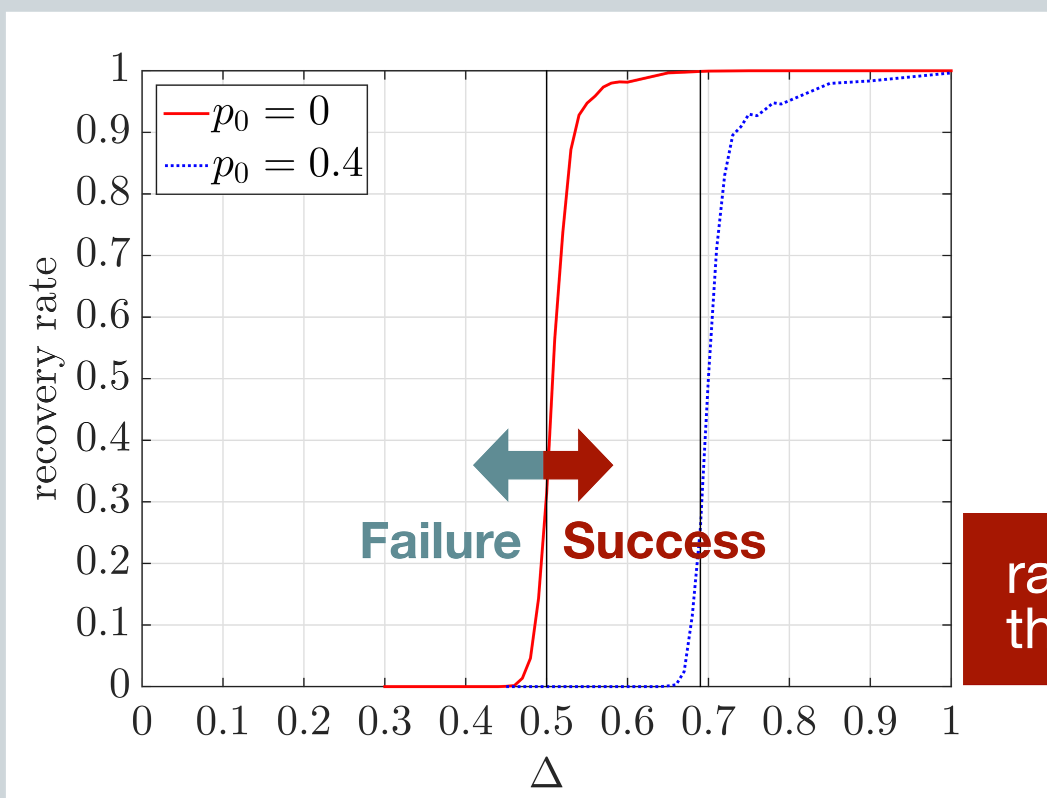

$\boldsymbol{b} \in \{0, \pm 1\}^N$
$\Pr(b_j = 0) = p_0$
$\Pr(b_j = +1) = (1 - p_0)/2$
$\Pr(b_j = -1) = (1 - p_0)/2$

## 4. Simulation Results




$\boldsymbol{b} \in \{0, \pm 1\}^N$
$p_0 = 0.2$
$\Delta = 0.7$

approach the theoretical performance as $N$ increases




$\boldsymbol{b} \in \{0, \pm 1\}^N$
$N = 1000$

rapidly increase around the theoretical boundary

[1] D. L. Donoho, A. Maleki, and A. Montanari, "Message-passing algorithms for compressed sensing," in Proc. Nat. Acad. Sci., vol. 106, no. 45, pp. 18914–18919, Nov. 2009.
[2] M. Nagahara, "Discrete signal reconstruction by sum of absolute values," IEEE Signal Process. Lett., vol. 22, no. 10, pp.1575–1579, Oct. 2015.